Let V be an irreducible projective variety of dim v.

Fix $A_0, \ldots, A_r \in \mathrm{Div}(V)$, and define

$$\mathcal{E} = \mathcal{O}(A_0) \oplus \ldots \oplus \mathcal{O}(A_r) \quad \text{and} \quad X = \mathbb{P}(\mathcal{E}), \quad L = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Then $\dim X = n = v + r$.

Idea: Choose A_i's so that $|L|$ has interesting behavior.

Lemma: Let $X = \mathbb{P}(\mathcal{E})$ as above.

(i.) $H^0\left(\mathcal{O}_X(k)\right) = \bigoplus_{a_0 + \ldots + a_r = k} H^0\left(\mathcal{O}_V\left(a_0 A_0 + \ldots + a_r A_r\right)\right)$

(ii.) $\mathcal{O}_X(1)$ is ample $\iff$ each A_i is ample on V.

(iii.) $\mathcal{O}_X(1)$ is nef $\iff$ each A_i nef.

(iv.) $\mathcal{O}_X(1)$ is big $\iff$ $a_0 A_0 + \ldots + a_r A_r$ is big for some
$$a_0, \ldots, a_r \in \mathbb{Z}_{\geq 0}$$

(v.) Given $m \in \mathbb{N}$, $\mathcal{O}_X(m)$ is base point free $\iff$ $|m A_i|$ is b.p.f.
$$\text{for each } i.$$

Pf Sketch:

(i.) Can check: $H^0\left(\mathcal{O}_X(k)\right) = H^0\left(S^k \mathcal{E}\right)$. (i.) follows easily.

(ii.) + (iii.) follow from properties of ample + nef v.b.s (See chapter 6 of Pos II.)

(iv.) Assume condition on A_i's holds.

Claim: $\#\left\{(a_0, \ldots, a_r) \mid \sum a_i = k \text{ and } \sum a_i A_i \text{ is big}\right\} \geq C \cdot k^r$

for some $C > 0$ and all $k \gg 0$.

Pf of Claim: Consider $\varphi: \mathbb{R}^{r+1} \to N^1(V)_{\mathbb{R}}$ defined
$$(b_0, \ldots, b_r) \longmapsto \sum b_i A_i$$

$\text{Big}(V) \subseteq N^1(V)_{\mathbb{R}}$ is an open convex cone, so $\varphi^{-1}(\text{Big}(V))$ is also.

The condition on the A_i's $\Rightarrow$ $\varphi^{-1}(\text{Big}(V))$ has nonempty intersection w/ the first orthant.

$\Rightarrow$ Each r-simplex $\Delta_t = \left\{\sum x_i = t, \, x_i \geq 0\right\} \subseteq \mathbb{R}^{r+1}$ meets $\varphi^{-1}\text{Big}(V)$

in a set Δ_t' of positive Lebesgue measure $\left(\text{of equal proportion } \beta \text{ to the entire simplex for each } t.\right)$

For $k \in \mathbb{N}$, $\#\ \mathbb{Z}$-points in $\Delta_k = \binom{k+r}{r} \geq c' \cdot k^r$, in k.

Thus, $\left(\text{very roughly}\right)$ for $k \gg 0$, $\#\ \mathbb{Z}$-points in $\Delta_k' \geq \beta C' k^r$,
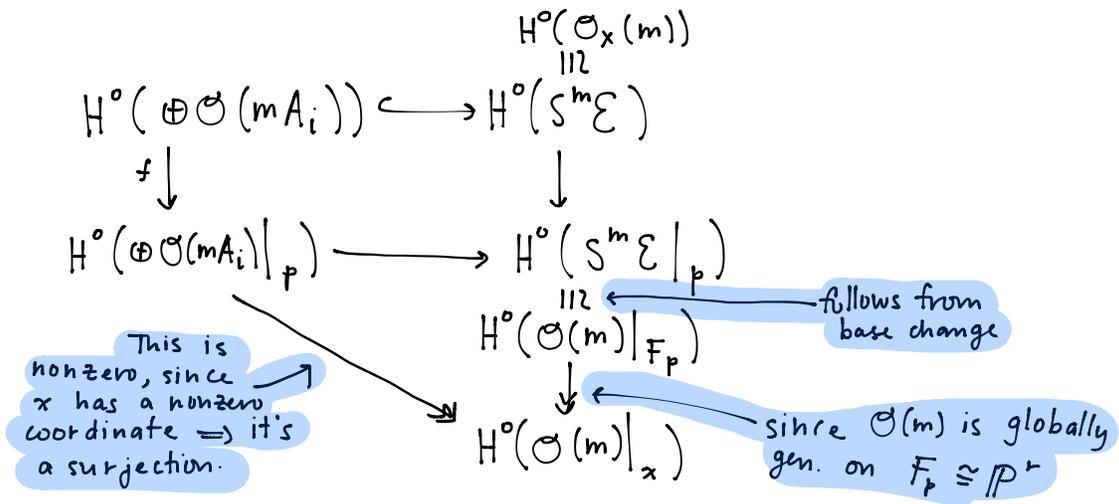
which was the claim.

Thus, in (i), $\geq C k^r$ of the summands are global sections of big divisors on V

Exercise: Use this to conclude that $h^0(\mathcal{O}_X(k)) \geq C'' k^{r+v}$ for

$k \gg 0$, and thus $\mathcal{O}_X(1)$ is big.

Exercise: Prove converse of (iv.)

(v.) Let $\pi : X \to V$ be the projection, and
$p \in V$, $F_p = \pi^{-1}(p)$, $x \in F_p$. Then we have

$$H^0(\mathcal{O}_X(m))$$
$$\shorteq$$
$$H^0(\oplus \mathcal{O}(mA_i)) \lhook\joinrel\longrightarrow H^0(S^m \mathcal{E})$$
$$f \downarrow \qquad\qquad\qquad \downarrow$$
$$H^0(\oplus \mathcal{O}(mA_i)|_p) \longrightarrow H^0(S^m \mathcal{E}|_p)$$
$$\shorteq$$
$$H^0(\mathcal{O}(m)|_{F_p})$$
$$\downarrow$$
$$H^0(\mathcal{O}(m)|_x)$$

This is nonzero, since x has a nonzero coordinate $\Rightarrow$ it's a surjection.

follows from base change

since $\mathcal{O}(m)$ is globally gen. on $F_p \cong \mathbb{P}^r$

If $\mathcal{O}(mA_i)$ is b.p.f. $\forall A_i$, then f is surjective,

so $H^0(\mathcal{O}_X(m)) \twoheadrightarrow H^0(\mathcal{O}(m)|_x) \quad \forall \ x \in X$

$\Rightarrow \mathcal{O}_X(m)$ is b.p.f.

Now suppose p is a base point of $\mathcal{O}(mA_1)$ (WLOG)

Then f is not surjective. In fact, $H^0(\oplus \mathcal{O}(mA_i))$
$$\downarrow$$
$$H^0(\mathcal{O}(mA_1)|_p)$$

is the zero map.

Thus, we can choose $x \in F_p$ w/ all but first "coordinate"

zero, and $H^0(\oplus \mathcal{O}(mA_i)) \oplus$ other summands of $\longrightarrow H^0(\mathcal{O}(m)|_x)$
$\phantom{zero, and H^0(\oplus \mathcal{O}(mA_i)) \oplus}$ $H^0(\mathcal{O}(m))$

is the zero map.

$\Longrightarrow H^0(\mathcal{O}(m)) \longrightarrow H^0(\mathcal{O}(m)|_x)$ is also the zero map, so

$\mathcal{O}(m)$ is $\underline{not}$ b.p.f. in this case. $\square$

$\boxed{\text{Apr 24}}$

$\underline{\text{Example}}$: Let V be a smooth projective curve of genus $g \geq 1$,

Let $A_0 = P$, a divisor of degree 0 and

$A_1 = A$, a divisor of degree $a > 0$.

So $X = \mathbb{P}(\mathcal{O}_V(P) \oplus \mathcal{O}_V(A))$ is a ruled surface.

Then P, A are nef and A is big, so $\mathcal{O}_X(1)$ is big + nef
by Lemma.

$\textcircled{a}$ $\overset{\text{Another}}{\text{example}}$ of big + nef but not f.g. (i.e. not semiample):

Take P nontorsion and $a = \deg A \gg 0$.

Then $H^0(\mathcal{O}_V(kP)) = 0 \; \forall k$. So, by the lemma,
$\mathcal{O}_X(k)$ is not globally generated.

Thus $\mathcal{O}_X(1)$ is not semiample, so it can't be f.g.

In fact, we can check this directly:

Let $R = R(\mathcal{O}_x(1)) = \oplus \, H^0(\mathcal{O}_x(k))$

Then $R = H^0\left(\text{sym}\left(\mathcal{O}_v(P) \oplus \mathcal{O}_v(A)\right)\right)$

Thus, we can think of R as a bigraded ring:

Set $R_{i,j} = H^0(\mathcal{O}_v(iP + jA)) \implies R_k = \underset{i+j=k}{\oplus} R_{i,j}$

Then for $k \geq 1$,

$$R_{k-1,1} = H^0(\mathcal{O}_v((k-1)P + A)) \subseteq R_k$$

This is nonzero, since A has large degree.
But for a, b s.t. $a + b = k-1$, $a \geq 0$, $b > 0$,

$$R_{a,1} \otimes R_{b,0} = 0, \text{ so}$$

$R_{k-1,1}$ consists of minimal generators of R for each k, so
R is **not** finitely generated.

(b.) In the above replace P w/ an e-torsion divisor.

Then $H^0(\mathcal{O}(k)) \neq 0 \; \forall \, k$, but

$\mathcal{O}_x(k)$ is b.p.f $\iff k \equiv 0 \pmod{e}$.

i.e. $M(X, \mathcal{O}_x(1)) \subsetneq \mathbb{N}(\mathcal{O}_x(1))$

semi-group
where $\mathcal{O}(k)$ is
b.p.f.

In fact, the base locus will be a section:

Every point on V is a base point of $\mathcal{O}_V(kP)$ but not of $\mathcal{O}_V(A)$

So on the fiber F_x, there will be a base point of $\mathcal{O}_x(k)$

Corresponding to the quotient $H^0(\mathcal{E}|_x) \twoheadrightarrow H^0(\mathcal{O}(P)|_x)$

(as in the proof of the lemma).

Ⓒ "Moving base point"

Let genus of $V = 1$, P deg 0, non torsion, $A = x$
for some $x \in V$.

Then, as in ⓐ, $|\mathcal{O}_x(k)|$ has a fixed component for every
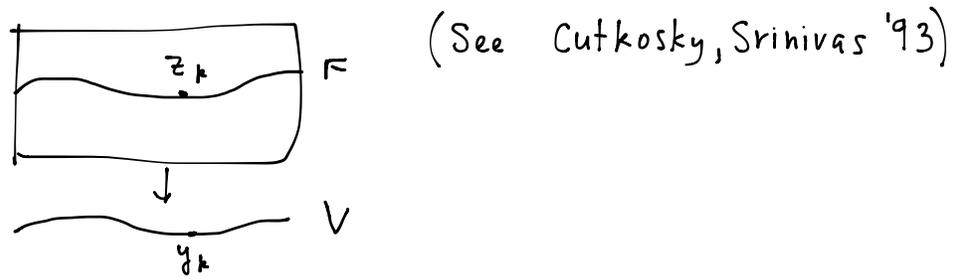$k \geq 1$. i.e.

$$|\mathcal{O}_x(k)| = F + |M_k|$$

Where M_k is the moving part (i.e. base locus contains
no divisors.), and F is the section corresponding to $\mathcal{E} \to \mathcal{O}_V(P)$.

For each $k \geq 1$, set $y_k \in V$ to be the unique point st.

$$y_k \stackrel{\equiv}{\scriptstyle\mathrm{lin}} (k-1)P + x$$

Then for $k \neq l$, $y_k \neq y_l$.

Exercise: $Bs|M_k| = z_k \in F$, the point in F over y_k.

(See Cutkosky, Srinivas '93)

(d.) Big line bundle w/ arbitrarily small volume:

Again let genus of $V = 1$. Fix $x \in V$, $a > 0$.

Let $A_0 = (1-a)x$, $A_1 = x$

Then $b_0 A_0 + b_1 A_1 = \left((b_0 + b_1) - b_0 a\right) x$

Thus, $H^0\left(\mathcal{O}_X(k)\right) = \bigoplus_{b=0}^{k} H^0\left(\mathcal{O}_V\left((k - ba)x\right)\right)$

$h^0\left(\mathcal{O}_V\left((k - ba)\right)\right) = \begin{cases} k - ba & \text{for } k \geq ba \quad \left(\lfloor \frac{k}{a} \rfloor \geq b\right) \\ 0 & \text{otherwise} \end{cases}$

So $h^0\left(\mathcal{O}_X(k)\right) = k + (k-a) + \ldots + \left(k - \lfloor \frac{k}{a} \rfloor a\right)$

$= k \cdot \lfloor \frac{k}{a} \rfloor - a \underbrace{\left(1 + 2 + \ldots + \lfloor \frac{k}{a} \rfloor\right)}_{\frac{\left(\lfloor \frac{k}{a} \rfloor + 1\right)}{2}}$

$\underset{\substack{\nearrow \\ \text{asymptotically}}}{=} \frac{k^2}{a} - \frac{k^2}{2a} + O(k) = \frac{k^2}{2a} + O(k)$

So $\text{vol}\left(\mathcal{O}_X(k)\right) = \lim_k \frac{\left(\frac{k^2}{2a}\right)}{k^2/2} = \frac{1}{a}$.

i.e. the volume can be arbitrarily small!

Example: Product of an elliptic curve:

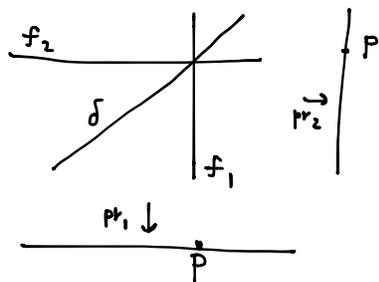Let E be a general elliptic curve and $V = E \times E$.

Let $pr_1, pr_2 : V \to E$ be the two projections

i) Nef cone calculation:

Fix $P \in E$. Then define in $N^1(V)_{\mathbb{R}}$:

$$f_1 = [\{P\} \times E], \quad f_2 = [E \times \{P\}], \quad \delta = [\Delta]$$

$\underset{\text{diagonal}}{\uparrow}$

Then
$$\delta \cdot f_1 = \delta \cdot f_2 = f_1 \cdot f_2 = 1 \quad \text{and}$$

$$(f_1)^2 = (f_2)^2 = 0$$

and $K_V \equiv 0$, so $\delta^2 = \delta \cdot (\delta + K_v) = 2g - 2 = 0$.

Thus, if $\delta = a_1 f_1 + a_2 f_2$, then $2a_1 a_2 = \delta^2 = 0$, so WLOG $a_1 = 0$

But $1 = \delta f_2 \neq a_2 f_2^2 = a_2$, so the three classes are lin.

independent, and, in fact, they span $N_1(V)_{\mathbb{R}}$.

Rmk: Since V is abelian, any effective curve is nef,
so $\overline{NE}(V) = Nef(V)$.

Claim: $\alpha \in N^1(x)_{\mathbb{R}}$ is nef $\iff$ $\alpha^2 \geq 0$, $\alpha \cdot h \geq 0$ for some ample h.

<u>Pf of claim</u>: If D is integral and H is ample and integral s.t. $D \cdot H > 0$ and $D^2 > 0$, then for $n \gg 0$, $nD \equiv_{lin}$ effective. (Use R-R for surfaces.)

Thus, taking limits, if the second statement holds, $\alpha \in \overline{NE}(V)$
$$= Nef(X).$$

"$\Rightarrow$" is clear. $\square$

So, in particular, take $h = f_1 + f_2 + \delta \subseteq int \ \overline{NE}(V) = Amp(V)$.

Then $\alpha = x f_1 + y f_2 + z \delta$ is nef $\Longleftrightarrow$
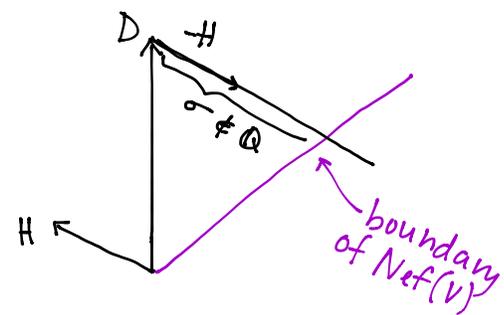
$$\tfrac{1}{2} \alpha^2 = xy + xz + yz \geq 0 \quad and$$
$$\tfrac{1}{2} \alpha \cdot h = x + y + z \geq 0$$

That is, $Nef(V) \subseteq N'(V)_\mathbb{R}$ is a circular cone.
(Apr 26)

ii) Cutkosky's Construction w/ $V = E \times E$

Choose D, H ample integral divisors $\in N'(V)_\mathbb{R}$ (abusing notation) s.t. the ray in $N'(V)_\mathbb{R}$ emanating from D in the direction of $-H$ meets the boundary of $Nef(V)$ at an irrational point.

boundary of Nef(V)

i.e. $\sigma := \max\{t \mid D - t \cdot H \text{ is nef}\} \notin \mathbb{Q}$

$\qquad = \text{smallest root of } (D - tH)^2$

(so most choices of D and H will give irrational σ.)

Apply Cutkosky's construction using

$\quad A_0 = D, \quad A_1 = -H.$

Then $\quad h^0(\mathcal{O}_X(k)) = \displaystyle\sum_{i+j=k} h^0(\mathcal{O}_V(iD - jH)).$

If $\dfrac{j}{i} < \sigma$, then $iD - jH$ is ample, so

$h^0(\mathcal{O}_V(iD - jH)) \underset{\uparrow}{=} \chi(\mathcal{O}_V(iD - jH)) = \frac{1}{2}(iD - jH)^2 + 1 - 1$

$\qquad\qquad$ Kodaira $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ arithmetic genus
$\qquad\qquad$ vanishing $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ of V
$\qquad\qquad$ $(K_V = 0)$ $\qquad\qquad\qquad\qquad = \frac{1}{2}(iD - jH)^2$

If $\dfrac{j}{i} > \sigma$, then $iD - jH \notin \overline{NE}(V)$, so it has no sections.

So $\quad h^0(\mathcal{O}_X(k)) = \displaystyle\sum_{\substack{i+j=k, \\ \frac{j}{i} < \sigma}} \frac{1}{2}(iD - jH)^2 \qquad \left(\frac{j}{i} < \sigma \implies \frac{k}{i} - 1 < \sigma \right.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \implies i > \frac{k}{\sigma+1} \right)$

$\qquad\qquad\qquad = \displaystyle\sum_{i > \frac{k}{1+\sigma}}^{k} \frac{1}{2}\left(iD - (k-i)H\right)^2$

Then $\quad \dfrac{h^0(\mathcal{O}_X(k))}{k^3/3!} = 3 \displaystyle\sum_{i \geq \frac{k}{1+\sigma}}^{k} \frac{1}{k} \cdot \left(\frac{i}{k}D - (1 - \frac{i}{k})H\right)^2$

RHS = Riemann sum $\left(w/\ \Delta x = \frac{1}{k} \right)$
for an integral of $q(x) := \left(x D - (1-x) H \right)^2$

Thus, taking the limit as $k \to \infty$ of both sides, we get

$$\text{Vol} \left(\mathcal{O}_X (1) \right) = 3 \int_{\frac{1}{1+\sigma}}^{1} q(x)\ dx$$

Since $\frac{1}{1+\sigma}$ is irrational, the volume will be irrational for most choices of D and H.

Explicit example of D, H:

Let $D = f_1 + f_2$, $H = 3f_2 + 3\delta$

Then σ is the smallest root of

$$s(t) = (D - t H)^2$$
$$= D^2 - 2t DH + t^2 H^2$$
$$= 2 - 18t + 18t^2$$

So $s(t) = 0 \implies t = \frac{3 \pm \sqrt{5}}{6} \implies \sigma = \frac{3 - \sqrt{5}}{6}$

$$q(x) = \left(x D - (1-x) H \right)^2 = D^2 x^2 - 2 x(1-x) DH + (1-x)^2 H^2$$

$$= 2x^2 - 18 \left(x - x^2 \right) + 18 \left(1 - 2x + x^2 \right)$$

$$= 38 x^2 - 54x + 18$$

$$\Rightarrow \text{vol}\left(\mathcal{O}_x(1)\right) = 3\int_{\frac{1}{1+\sigma}}^{1}\left(38x^2 - 54x + 18\right)dx$$

$$= \frac{135\sqrt{5} - 77}{722} \approx 0.31145$$